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INTERFACE DESIGN OF NEUTRAL ELASTIC INCLUSIONS

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Abstract—The paper studies the design of the neutral elastic inclusions that do not disturb the prescribed uniform stress field in the surrounding elastic body. Such neutral inclusion does not exist if a perfectly bonded interface between inclusion and elastic body is assumed. The design method presented here is based on the model of imperfect interface characterized by the interface parameters.

The basic equations for interface design of a single neutral elastic inclusion in plane and anti-plane deformations are given and the corresponding interface parameters are discussed for several typical inclusion shapes. In particular, the design of a neutral inclusion under uniaxial or equal-biaxial tension is studied with a restriction imposed on the interface parameters that facilitates implementation of the designed interface. The results obtained in the paper affirm the feasibility of designing a neutral elastic inclusion in many typical cases by means of the imperfect interface.

1. INTRODUCTION

It is commonly believed that a hole made in an elastic body will inevitably disturb the original stress field and often lead to a stress concentration. Mansfield (1953) is one of the first who recognized the feasibility of designing a reinforced "neutral" hole which does not alter the original stress distribution in the cut elastic body. For related works, see e.g. Savin (1961), Cherepanov (1974), Bjorkman and Richards (1976, 1979), Richards and Bjorkman (1982), Wheerler (1992), Budiansky *et al.* (1993) and Senocak and Waas (1993, 1995, 1996).

The analogous problem of a neutral elastic inclusion, which does not cause any stress disturbance in the surrounding elastic body, has not received deserved attention despite its importance for various problems in the design of composite materials and structures. This can be attributed to, to some extent, the non-existence of such neutral elastic inclusion when a conventional perfectly bonded interface between inclusion and elastic body is presumed (it will be shown below). The design method proposed here is based on the model of imperfect interface across which tractions are continuous and jumps in displacement are proportional to their respective traction components in terms of the interface parameters. This model has originally been proposed to describe the imperfectly bonded interfaces appearing in various composite materials and structures, see, for example, Benveniste (1984), Achenbach and Zhu (1989, 1990), Hashin (1990, 1991), Pagano and Tandon (1990), Thorpe and Jasiuk (1992) and Jun and Jasiuk (1993). The present paper will make its profitable application to the design of neutral elastic inclusions.

Consider a homogeneous elastic body, finite or infinite in extent and simply or multiply connected, undergoing a uniform stress state under the prescribed loading system. Assume that the elastic body is now cut out over a number of simply connected sub-domains and filled up with some homogeneous elastic inclusions. The problem raised in the paper is how to design the interfaces between inclusions and elastic body such that the embedded inclusions are "neutral" in the sense that they do not disturb the original uniform stress field in the cut elastic body. In other words, the concept of a neutral inclusion defined here emphasizes the undisturbed stress state *outside* the inclusion (as will be seen below, it implies the uniformity of stress state inside the inclusion for the present problems). This is obviously different from the "equal-strain inclusion" in the sense of Eshelby (Eshelby (1957)), which usually destroys the uniformity of stress field outside the inclusion and then is not "neutral"

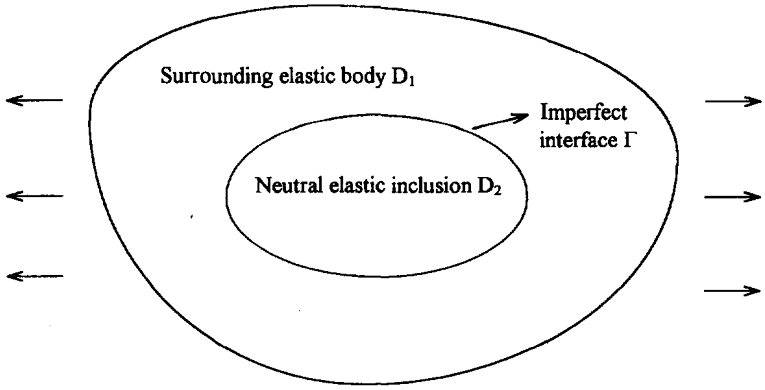

Fig. 1. The design of a neutral elastic inclusion embedded in an elastic body in order that the resulting composite is elastically equivalent to a homogeneous body under the prescribed uniform stress field.

(see Eshelby (1959), Sendeckji (1970) and Ru and Schiavone (1996)). It is believed that the concept of neutral elastic inclusion will find its applications in many practical problems where the stress concentration caused by material mismatch is a matter of utmost concern.

Since this problem for multiple inclusions reduces to the single inclusion problems for each of the embedded inclusions, the paper focuses on the design of a single neutral elastic inclusion. Throughout the text, $z = x+iy = re^{i\theta}$ denotes the complex coordinate, and D_2 and D_1 represent the domains occupied by the elastic inclusion and the cut elastic body, respectively. The interface between D_2 and D_1 is denoted by Γ, see Fig. 1. The subscripts 1 and 2 will refer to the domains D_1 and D_2, respectively.

2. ANTI-PLANE SHEAR

First, we consider the neutral elastic inclusion in anti-plane shear. The anti-plane displacement $w(z)$ satisfies the harmonic equilibrium equation, and the interface conditions along Γ are given by

$$h(z)(w_1 - w_2) = \mu_1 \frac{\partial w_1}{\partial N} = \mu_2 \frac{\partial w_2}{\partial N} \tag{1}$$

where μ is the shear modulus, N denotes the direction of the outward normal to Γ, and $h(z)$ is an interface parameter. In particular, $h(z) = 0$ represents the traction-free boundary, and $h(z) =$ infinity corresponds to a perfectly bonded interface. The interface model (1) can be realized in practice by the adhesive layer which bonds the inclusion to the elastic body (for example, an elastic interphase layer, see Hashin (1991), or continuously distributed linear springs, see Gecit and Erdogan (1978) and Achenbach and Zhu (1989, 1990)). In doing so, $h(z)$ should be inversely proportional to the thickness, or directly proportional to the density of the adhesive layer, then it can be designed arbitrarily by controlling the latter. The only restriction is that $h(z)$ must be non-negative everywhere.

For convenience, we introduce the analytic functions $\chi_1(z)$ and $\chi_2(z)$, whose real parts give $\mu_1 w_1(z)$ and $\mu_2 w_2(z)$, in D_1 and D_2, respectively. Hence, (1) can be written in a complex form

$$\chi_1(z) = \delta\chi_2(z) + (\delta - 1)\overline{\chi_2(z)} + \frac{\mu_1}{2h(z)}[\chi_1'(z)e^{iN(z)} + \overline{\chi_1'(z)}e^{-iN(z)}] \tag{2}$$

where $e^{iN(z)}$ denotes (in complex form) the outward unit normal to Γ, and

$$\delta = \frac{\mu_1 + \mu_2}{2\mu_2}. \tag{3}$$

Let the prescribed uniform stress field be characterized by $\chi_1(z) = Cz$, where C is a given complex number. According to the definition of a neutral inclusion, the original uniform stress field in the cut elastic body remains undisturbed when the neutral inclusion is inserted, then we have $\chi_1(z) = Cz$ in D_1. Hence, for a neutral inclusion, (2) is reduced to

$$Cz = \delta\chi_2(z) + (\delta - 1)\overline{\chi_2(z)} + \frac{\mu_1}{2h(z)}[Ce^{iN(z)} + \bar{C}e^{-iN(z)}] \tag{4}$$

whose imaginary part gives

$$Cz - \overline{Cz} = \chi_2(z) - \overline{\chi_2(z)}. \tag{5}$$

This condition (5) determines uniquely $\chi_2(z)$ to within an arbitrary real number C_0

$$\chi_2(z) = Cz + C_0. \tag{6}$$

Substituting (6) into the interface condition (4) yields

$$(1-\delta)(Cz + \overline{Cz}) = (2\delta - 1)C_0 + \frac{\mu_1}{2h(z)}[Ce^{iN(z)} + \bar{C}e^{-iN(z)}]. \tag{7}$$

This equation relates the interface parameter $h(z)$ to the shape of neutral elastic inclusion. For existence of the neutral inclusion, arbitrary real number C_0 should be properly chosen to eliminate the possible rigid body translation (caused by the choice of the coordinates) between the inclusion and elastic body. In particular, if the inclusion has two mutually orthogonal axes of symmetry, we choose them as the coordinate axes, and then $C_0 = 0$.

The eqn (7) can not hold for any interface curve if $h(z)$ is infinite everywhere. This means that there is no neutral elastic inclusion if a conventional perfectly bonded interface is presumed. This result suggests that the concept of imperfect interface plays an indispensable role in the design of neutral elastic inclusions.

2.1. *Determination of the inclusion shape when* h(z) *is given*

First, we apply (7) to the determination of the neutral inclusion shape when the variation of $h(z)$ along the interface is prescribed.

Homogeneous interface ($h(z)$ is a constant). We choose the coordinates such that C is a real number. Due to the symmetries of the problem, the inclusion can be assumed to be symmetric about the rectangular axes and then $C_0 = 0$. The eqn (7) has now the form

$$\cos[N(z)] = \frac{2h(1-\delta)x}{\mu_1}. \tag{8}$$

It turns out that the neutral inclusion is a circle with the radius R given by

$$\frac{\mu_1}{2Rh} = 1 - \delta = \frac{\mu_2 - \mu_1}{2\mu_2}. \tag{9}$$

This result is independent of the prescribed uniform stress field. Hence, if the interface is restricted to be homogeneous, the neutral elastic inclusion exists only when it is "harder" than the surrounding elastic body (namely, $\mu_2 > \mu_1$). In such a case, the only neutral inclusion is a circle with the radius R given by (9). For example, for a neutral *rigid* inclusion, (9) reduces to: $\mu_1 = Rh$. Hence, a rigid circular inclusion with the interface parameter h

given by this formula will not disturb any uniform stress field in the surrounding elastic body.

C *is a real number and* h(z) = h(x). It is the case when the unknown inclusion has two mutually orthogonal axes of symmetry, one of which is parallel to the direction of the prescribed uniform shear stress and $h(z)$ depends on the distance from z to the other. In this case, the coordinates can be chosen such that the inclusion is symmetric about the rectangular axes, then $C_0 = 0$. If we denote the unknown curve Γ by $y = y(x)$, the eqn (7) gives

$$\frac{\pm y'(x)}{\sqrt{1+[y'(x)]^2}} = \frac{2h(x)(1-\delta)x}{\mu_1}. \tag{10}$$

It determines the shape of neutral inclusion when $h(x)$ is given. For example, if $h(x)$ is inversely proportional to the absolute value of x, the neutral inclusion is a rhombus symmetric about the coordinate axes.

2.2. *Interface parameter* h(z) *of a neutral inclusion of given shape*

Next, we derive the interface parameter $h(z)$ of a neutral elastic inclusion when its shape is given. Here, we are only interested in the inclusion shape which is symmetric about two mutually orthogonal axes. Choose these axes of symmetry as the coordinate axes, then $C_0 = 0$. Furthermore, let

$$C = |C|e^{i\theta_0}\left(\frac{\pi}{2} \geqslant \theta_0 > -\frac{\pi}{2}\right),$$

eqn (7) gives

$$h(z) = \frac{\mu_1 \cos[N(\theta)+\theta_0]}{2(1-\delta)r\cos[\theta+\theta_0]}. \tag{11}$$

It is assumed that $1 > \delta(\mu_2 > \mu_1)$.

Elliptic inclusion. If the elliptic inclusion (as shown in Fig. 2(b)) does not degenerate to a circular one, $N(\theta) = \theta$ only at the four points of symmetry on Γ. Consequently, from (11), $h(z)$ is non-negative on Γ when and only when C is a real or a pure imaginary number. Hence, the design of an elliptic neutral inclusion is available, with the present method, only when the prescribed uniform shear stress is parallel to one of its two principle axes. Note that for an elliptic curve shown in Fig. 2(b)

$$e^{iN(z)} = \frac{\left(1+\frac{a^2}{b^2}\right)z+\left(1-\frac{a^2}{b^2}\right)\bar{z}}{2a\sqrt{1-\frac{a^2}{b^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)y^2}}. \tag{12}$$

Then, if C is a positive real number, the associated parameter $h(z)$ is given by

$$h(z) = \frac{\mu_1}{2a(1-\delta)\sqrt{1+\frac{a^2}{b^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2}}. \tag{13}$$

Similar results can be drawn for an arbitrary smooth convex inclusion with two mutually orthogonal axes of symmetry. Hence, the present design method is applicable for a wide class of neutral elastic inclusion in anti-plane shear.

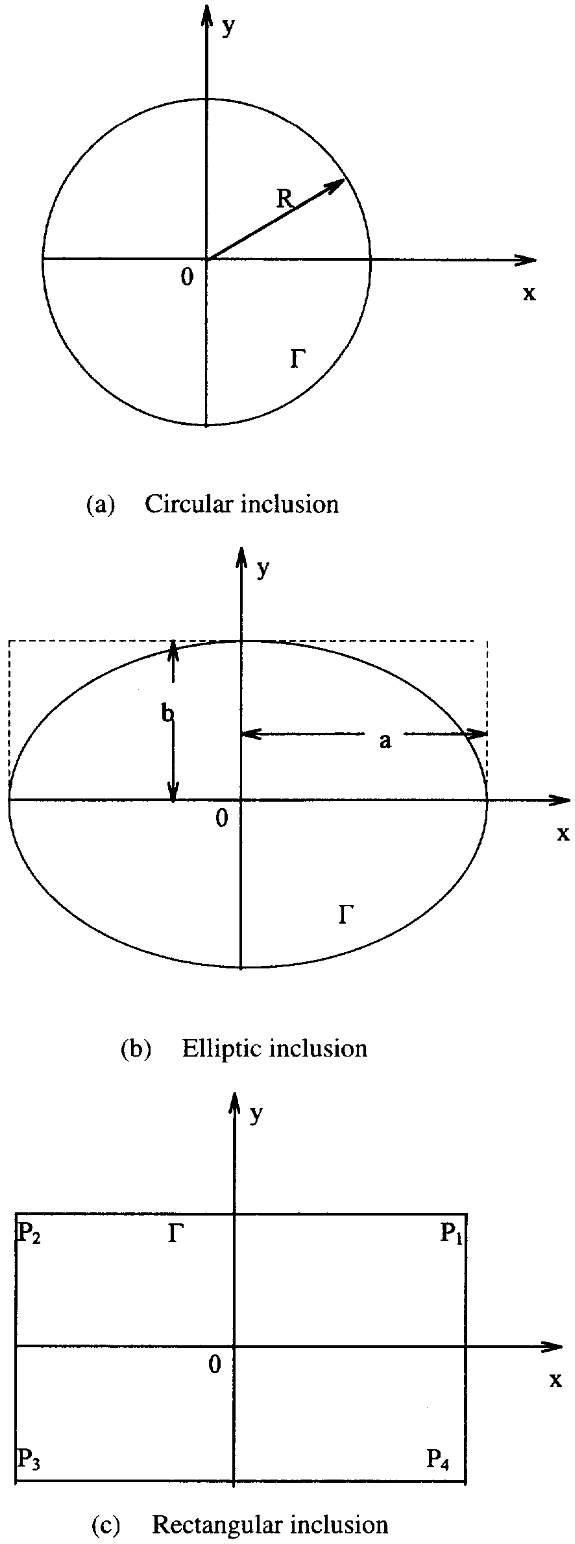

Fig. 2. Several typical inclusion shapes (a) circular inclusion; (b) elliptic inclusion; (c) rectangular inclusion.

Rectangular inclusion. According to the above analysis, if the prescribed uniform shear stress is parallel to one of the four edges, $h(z)$ given by (11) is non-negative for a rectangular neutral inclusion, shown in Fig. 2(c). On the other hand, if it is not the case (that is, if θ_0 is not equal to 0 or $\pi/2$), $\cos[N(\theta)+\theta_0]$ is constant on each of the four edges and changes in sign only at the corners P_1 and P_3 (if $\theta_0 > 0$), or at P_2 and P_4 (if $\theta_0 < 0$). Hence, in order that $h(z)$ is non-negative on Γ, $\cos[\theta+\theta_0]$ is required to change its sign at the same corners as $\cos[N(\theta)+\theta_0]$ does. This leads to

$$\arctan\frac{|P_4P_1|}{|P_1P_2|}+|\theta_0| = \frac{\pi}{2} \tag{14}$$

which guarantees the positivity of $h(z)$. For example, for a square inclusion, $h(z)$ is positive if the prescribed uniform shear stress is parallel to one of its two diagonals.

The results obtained in this section indicate that the model (1) of imperfect interface can be used effectively in the design of a wide class of neutral elastic inclusions undergoing anti-plane shear. In practice, an arbitrary adhesive layer (such as an elastic interphase layer or a layer of distributed joints) can serve as such an imperfect interface. In this way, the required interface parameter $h(z)$ can be accomplished (at least approximately) by adjusting the thickness of adhesive layer or the density of distributed joints (see Gecit and Erdogan (1978) and Hashin (1991)). In particular, interface design derived by the present method is independent on the magnitude of the prescribed uniform stress field.

3. PLANE DEFORMATIONS

Now, we consider the neutral elastic inclusion in plane deformations. It is well-known that the stresses and displacements can be given in terms of two analytic functions $\phi(z)$ and $\psi(z)$ (see Muskhelishvili (1965))

$$2\mu(u_x+iu_y) = [\kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}] \tag{15}$$

$$\sigma_{xx}+\sigma_{yy} = 2[\phi'(z)+\overline{\phi'(z)}], \quad \sigma_{xx}-i\sigma_{xy} = \phi'(z)+\overline{\phi'(z)}-[\bar{z}\phi''(z)+\psi'(z)]. \tag{16}$$

Here, $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, and ν is Poisson's ratio. Consequently, the boundary tractions and displacements are given by

$$2\mu(u_n+iu_t) = e^{-iN(z)}[\kappa\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}] \tag{17}$$

$$\sigma_{nn}-i\sigma_{nt} = \phi'(z)+\overline{\phi'(z)}-e^{2iN(z)}[\bar{z}\phi''(z)+\psi'(z)]. \tag{18}$$

Across the interface Γ, the inclusion is assumed to be bonded to the cut elastic body by an imperfect interface described in Section 1, in terms of the normal and tangential interface parameters $m(z)$ and $n(z)$, as follows

$$[[\sigma_{nn}-i\sigma_{nt}]] = 0 \tag{19}$$

$$\sigma_{nn} = m(z)[[u_n]], \quad \sigma_{nt} = n(z)[[u_t]] \tag{20}$$

where $[[*]] = (*)_1-(*)_2$ denotes the jump across Γ. When $m(z) = n(z) = 0$, (19–20) represent the traction-free boundary conditions; and if $m(z) = n(z) =$ infinity, (19–20) correspond to a perfectly bonded interface. Similar to anti-plane shear, the interface model (19–20) can be realized in practice using an adhesive layer. In doing so, any one of the two interface parameters, $m(z)$ and $n(z)$, or a combination of them, can be specified at will by controlling the thickness or the density of adhesive layer. However, if only a single adhesive material is used, the ratio $m(z)/n(z)$ is usually a material constant, independent of the thickness of the adhesive layer. For example, the ratio $m(z)/n(z)$ can be assumed to be unity for a layer of distributed springs (see Achenbach and Zhu (1989, 1990)), or be a certain constant larger than one for an elastic interphase layer (see Hashin (1991)). Hence, in order that the designed interface can be easily implemented using a single adhesive material, interface design of a neutral inclusion should be carried out under the restriction that the ratio m/n is a constant not less than one (or equivalently, the ratio $(m-n)/(m+n)$ is a non-negative constant not larger than one). In Section 3.2, the uniaxial (and equal-biaxial) tension in plane deformations will be studied in detail with such a restriction condition.

However, the above analysis does not imply that the interface with two independent parameters $m(z)$ and $n(z)$ should be precluded. For instance, to design a neutral inclusion of given shape (see Section 3.1), $m(z)$ and $n(z)$ must be designed independently. In this case, more than one adhesive material is needed. For example, combining an elastic interphase layer with a continuous distribution of joints (such as springs), an imperfect interface with two independently varying parameters $m(z)$ and $n(z)$ can be actually constructed. In Section 3.1, we discuss the design of a neutral inclusion of given shape and reduce it to the problem of designing an interface of two given independent parameters $m(z)$ and $n(z)$.

Now, to derive the counterpart of eqn (7) to plane deformations, let the prescribed uniform stress field be characterized by

$$\phi_1(z) = Az, \quad \psi_1(z) = Bz \tag{21}$$

where A and B are given real and complex numbers, respectively. For a neutral elastic inclusion, the condition (19) gives

$$2A\bar{z}+Bz = \overline{\phi_2(z)}+\bar{z}\phi_2'(z)+\psi_2(z). \tag{22}$$

According to the uniqueness of the solution of traction boundary problem in plane elasticity, (22) determines uniquely $\phi_2(z)$ and $\psi_2(z)$ in D_2 apart from the additional terms A_0z and B_0, representing a rigid-body displacement, then

$$\phi_2(z) = (A+iA_0)z, \quad \psi_2(z) = Bz+B_0 \tag{23}$$

where A_0 is an arbitrary real number and B_0 an arbitrary complex number. The remaining interface condition (23) is now of the form

$$\begin{aligned} 2A-Be^{2iN(z)} = {} & \frac{m(z)+n(z)}{4}\left\{\frac{e^{iN(z)}}{\mu_1}[(\kappa_1-1)A\bar{z}-Bz]\right. \\ & \left.-\frac{e^{iN(z)}}{\mu_2}[(\kappa_2-1)A\bar{z}-Bz-i(\kappa_2+1)A_0\bar{z}-B_0]\right\} \\ & +\frac{m(z)-n(z)}{4}\left\{\frac{e^{-iN(z)}}{\mu_1}[(\kappa_1-1)Az-\overline{Bz}]\right. \\ & \left.-\frac{e^{-iN(z)}}{\mu_2}[(\kappa_2-1)Az-\overline{Bz}+i(\kappa_2+1)A_0z-\overline{B_0}]\right\}. \end{aligned} \tag{24}$$

This equation governs the interface design of a neutral elastic inclusion in plane deformations. For existence of the neutral inclusion, arbitrary real number A_0 and complex number B_0 should be properly chosen to eliminate the possible rigid body displacement between the inclusion and elastic body. In particular, if the inclusion is geometrically symmetric about two mutually orthogonal axes and the latter is chosen as the coordinate axes, we have $A_0 = B_0 = 0$.

If the interface is perfectly bonded and then both $m(z)$ and $n(z)$ are infinite, eqn (24) reduces to

$$\frac{1}{\mu_1}[(\kappa_1-1)A\bar{z}-Bz]=\frac{1}{\mu_2}[(\kappa_2-1)A\bar{z}-Bz-i(\kappa_2+1)A_0\bar{z}-B_0] \quad (25)$$

which fails for any interface curve unless two materials comprising the inclusion and elastic body, respectively, are identical. Hence, there is no neutral elastic inclusion when a conventional perfect interface is assumed.

3.1. *Interface parameters of a neutral inclusion of given shape*

First, we apply (24) to derive the interface parameters $m(z)$ and $n(z)$ for a neutral inclusion when its shape is given. Here, we are only interested in the inclusion which has two mutually orthogonal axes of symmetry. Choose the axes of symmetry as the coordinate axes, then $A_0 = B_0 = 0$. To obtain the expressions for $m(z)$ and $n(z)$, respectively, it is convenient to write (24) in the forms

$$2\frac{4A-Be^{2iN(z)}-\bar{B}e^{-2iN(z)}}{m(z)}=A\eta[e^{iN(z)}\bar{z}+e^{-iN(z)}z]-\lambda[Be^{iN(z)}z+\overline{Bz}e^{-iN(z)}] \quad (26)$$

$$2\frac{Be^{2iN(z)}-\bar{B}e^{-2iN(z)}}{n(z)}=A\eta[ze^{-iN(z)}-\bar{z}e^{iN(z)}]+\lambda[Be^{iN(z)}z-\overline{Bz}e^{-iN(z)}] \quad (27)$$

where

$$\lambda=\frac{1}{\mu_1}-\frac{1}{\mu_2},\quad \eta=\frac{\kappa_1-1}{\mu_1}-\frac{\kappa_2-1}{\mu_2}. \quad (28)$$

It is noted that $\kappa = 2$ when $\nu = 1/4$ for plane strain or when $\nu = 1/3$ for plane stress. Since the Poisson's ratio ν lies between 1/4 and 1/3 for most of the practical materials, the two bi-material constants defined by (28) can be assumed to have the same sign.

Circular inclusion. First we consider a circular neutral inclusion. Choose the coordinates such that A and B are two real numbers. Thus, (26) and (27) give

$$\frac{B}{n(z)}=\lambda\frac{RB}{2} \quad (29)$$

$$\frac{2}{m(z)R}=\frac{\eta A-\lambda B\cos[2\theta]}{2A-B\cos[2\theta]}. \quad (30)$$

Obviously, the ratio m/n is not a constant on Γ unless $A = 0$ or $B = 0$. When $B = 0$ (equal-biaxial tension), n is arbitrary (see (29)) and m is given by

$$\frac{4}{mR}=\eta. \quad (31)$$

On the other hand, if $A = 0$ (pure shear), the circular neutral inclusion is available when $m = n$ and

$$\frac{2}{mR}=\lambda. \quad (32)$$

In order to avoid interpenetration of materials, the negative normal displacement jump given by the solution is not allowed to exceed the thickness of interface layer, see Hashin (1991). Finally, for the uniaxial tension along the x-axis (then $2A = -B > 0$), n and $m(z)$ are given by (29) and

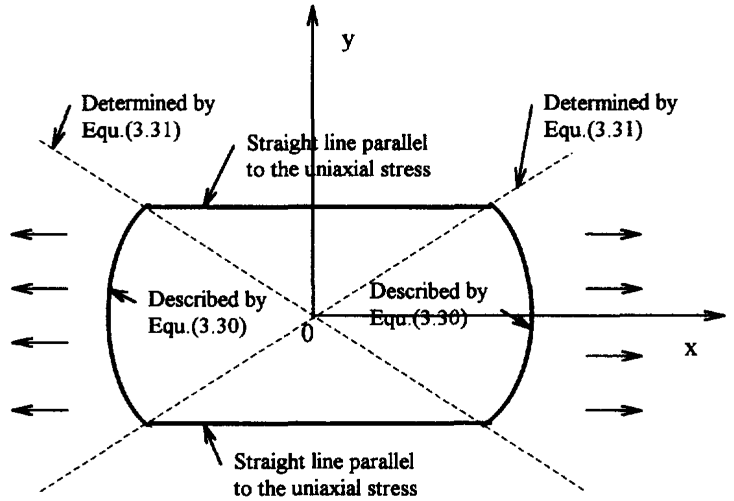

Fig. 3. Interface design of a neutral elastic inclusion under uniaxial tension when the ratio $m-n/m+n$ is a given constant along the interface.

$$\frac{2}{m(z)R} = \frac{\frac{\eta}{2}+\lambda\cos[2\theta]}{1+\cos[2\theta]}, \tag{33}$$

respectively. It is noted that $m(z)$ may be negative on two arcs of Γ which, if any, are determined by the condition

$$\frac{\eta}{2}+\lambda\cos[2\theta] < 0.$$

This implies that, when $(\eta-2\lambda) < 0$, a perfectly circular neutral inclusion under uniaxial tension is unavailable with the present method. (A similar circumstance was encountered in the design of neutral holes, see Mansfield (1953), where the positivity of the section-area or the modulus of reinforcement disqualifies some arcs of curve for the boundary of a neutral hole.) Instead, as will be seen below, the two disqualified arcs, on which $m(z)$ given by (33) is negative, can be replaced by two straight lines parallel to the uniaxial tensile stress (cf. Fig. 3).

Rectangular inclusion. For a rectangular neutral inclusion, as shown in Fig. 2(c), the interface parameters $m(z)$ and $n(z)$ are obtained from (26–27) as

$$\frac{\pm 2B_2}{n(z)} = A\eta y+\lambda[xB_2+yB_1], \quad \frac{\pm 2(2A-B_1)}{m(z)} = A\eta x+\lambda[B_2 y-xB_1] \tag{34}$$

for two vertical edges (where the positive sign "+" is taken on the left-hand side of (34) for P_4P_1 and the negative sign "−" is taken for P_2P_3), and

$$\frac{\pm 2B_2}{n(z)} = A\eta x+\lambda[yB_2-xB_1], \quad \frac{\pm 2(2A+B_1)}{m(z)} = A\eta y+\lambda[xB_2+yB_1] \tag{35}$$

for two horizontal edges (where the positive sign "+" is taken on the left-hand side of (35) for P_1P_2 and the negative sign "−" is taken for P_3P_4), where $B = B_1+iB_2$. When $B_2 = 0$, we have that $n = 0$ on the whole interface Γ and m is constant on each of all four edges. In particular, m is positive under equal-biaxial tension provided that $\eta > 0$. On the other hand, when $A = B_1 = 0$, $m = 0$ on the whole interface Γ and n is constant on each of all four edges. In this case, n is positive when and only when $\lambda > 0$. Hence, the design of neutral rectangular inclusion under typical loading conditions reduces to the problem of designing an interface with $m = 0$ or $n = 0$.

Of particular interest is the uniaxial tension (along the x-axis, then $2A = -B_1 > 0$, $B_2 = 0$). Assuming that $(\eta - 2\lambda) \neq 0$, it is found that $m = n = 0$ on the two edges parallel to the uniaxial tensile stress and

$$n = 0, \quad \frac{8}{m} = (2\lambda + \eta)|x| \tag{36}$$

on the other two edges (normal to the tensile force). Here, m given by (36) is positive when and only when $2\lambda + \eta > 0$. A consequence of these results is that any straight line parallel to the uniaxial tensile stress is qualified as a part of the boundary of a neutral elastic inclusion (see Fig. 3) if it is endowed with the null interface parameters $m = n = 0$. It can be proved from (26, 27) that this is a characteristic property of the uniaxial tension. This result is very useful in the design of neutral elastic inclusion under uniaxial tension (see Section 3.2). In doing so, a narrow gap may be made along this straight-line interface such that the possible negative normal displacement jump, caused by the Poisson contraction, does not lead to interpenetration of materials.

Elliptic inclusion. Finally we consider an elliptic neutral inclusion under an arbitrary bi-axial tension (then A and B are real numbers). In this case, (26–27) give

$$\begin{aligned} \frac{4A - 2B\cos[2N(\theta)]}{m(z)} &= Ar\eta\cos[N(\theta) - \theta] - Br\lambda\cos[N(\theta) + \theta] \\ \frac{2B\sin[2N(\theta)]}{n(z)} &= Ar\eta\sin[\theta - N(\theta)] + Br\lambda\sin[N(\theta) + \theta]. \end{aligned} \tag{37}$$

In particular, for equal-biaxial tension, (37) and (12) give

$$n = 0, \quad \frac{8}{m(z)} = \eta r^2 \frac{\left(1 + \dfrac{a^2}{b^2}\right) + \left(1 - \dfrac{a^2}{b^2}\right)\cos[2\theta]}{a\sqrt{1 - \dfrac{a^2}{b^2}\left(1 - \dfrac{a^2}{b^2}\right)\dfrac{y^2}{a^2}}}. \tag{38}$$

Hence, $m(z)$ is positive on Γ if and only if $\eta > 0$. Next, for the uniaxial tension parallel to the major axis (x-axis), we find

$$\begin{aligned} 4\frac{1 + \cos[2N(\theta)]}{m(z)} &= \frac{r^2}{a\sqrt{1 + \dfrac{a^2}{b^2}\left(\dfrac{1}{b^2} - \dfrac{1}{a^2}\right)y^2}}\left[(\eta + 2\lambda)\cos^2[\theta] + (\eta - 2\lambda)\frac{a^2}{b^2}\sin^2[\theta]\right] \\ 4\frac{\sin[2N(\theta)]}{n(z)} &= \frac{r^2}{a\sqrt{1 + \dfrac{a^2}{b^2}\left(\dfrac{1}{b^2} - \dfrac{1}{a^2}\right)y^2}}\left[(\eta + 2\lambda)\frac{a^2}{b^2} - (\eta - 2\lambda)\right]\sin[\theta]\cos[\theta] \end{aligned} \tag{39}$$

and, for the uniaxial tension along the minor axis (y-axis), we have

$$4\frac{1-\cos[2N(\theta)]}{m(z)} = \frac{r^2}{a\sqrt{1+\frac{a^2}{b^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2}}\left[(\eta-2\lambda)\cos^2[\theta]+(\eta+2\lambda)\frac{a^2}{b^2}\sin^2[\theta]\right]$$

$$4\frac{\sin[2N(\theta)]}{n(z)} = \frac{r^2}{a\sqrt{1+\frac{a^2}{b^2}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2}}\left[(\eta+2\lambda)-(\eta-2\lambda)\frac{a^2}{b^2}\right]\sin[\theta]\cos[\theta]. \quad (40)$$

Obviously, the positivity of interface parameters depends on the material and geometric constants. The sufficient conditions for the positivity of $n(z)$ given by (39) and (40) are

$$2\lambda \pm \eta > 0. \quad (41)$$

On the other hand, for any one of (39) and (40), there are two symmetric arcs of Γ on which $m(z)$, given by (39) or (40), is negative. As a result, the design of a perfect elliptic neutral inclusion is unavailable with the present method. In order to achieve a closed inclusion boundary, the two disqualified arcs should be replaced by two straight lines parallel to the uniaxial tensile stress, respectively.

From the results obtained in this section, it is seen that the design of a neutral inclusion of given shape often requires that the two interface parameters $m(z)$ and $n(z)$ can be specified independently. In principle, such an interface can be constructed using two or more different adhesive materials. Of course, the implementation of such a combined adhesive interface is technically more complicated.

3.2. *Interface design when the ratio* (m − n)/(m + n) *is a given constant*

In this section we study the interface design of a neutral inclusion under the restriction that the ratio $(m-n)/(m+n)$ is a constant along the interface. This restriction guarantees that the designed interface can be easily implemented using a single adhesive material.

Equal-biaxial tension. From Section 3.1, the circular inclusion with a constant parameter m, given by (31), can serve as a neutral inclusion under equal-biaxial tension. In fact, the circular inclusion is the only neutral inclusion under equal-biaxial tension if the ratio $(m-n)/(m+n)$ is a constant on Γ. To see this, note that the right-hand side of (24) is real for equal-biaxial tension, then (24) gives

$$(1-\omega)(e^{iN(z)}\bar{z}-e^{-iN(z)}z) = 0. \quad (42)$$

It follows that the neutral inclusion must be a circle.

Uniaxial tension. Now we examine the uniaxial tension. In this case, the unknown inclusion should be symmetric about two mutually orthogonal axes, one of which is parallel to the uniaxial tensile stress. We choose the axes of symmetry as the coordinate axes such that the x-axis is parallel to the uniaxial tensile stress. Thus, we have $2A = -B > 0$ and $A_0 = B_0 = 0$. Hence, the following combination

$$\frac{2Ae^{-iN(z)}-Be^{iN(z)}}{m(z)+n(z)}$$

is a real quantity. Then, it follows from (24) that

$$(\eta-2\lambda)(z-\bar{z})+\omega[e^{2iN(z)}(2\lambda z+\eta\bar{z})-e^{-2iN(z)}(2\lambda\bar{z}+\eta z)] = 0. \quad (43)$$

If $m(z) = n(z)$ (then $\omega = 0$) on Γ, the neutral inclusion exists only when $2\lambda = \eta$. Hence, in what follows, we assume that the ratio $(m-n)/(m+n)$ is a non-zero constant ω. In this case, it is found from (43) that

$$e^{2iN(z)} = \frac{-i(\eta-2\lambda)\sin[\theta] \pm \sqrt{\omega^2|(2\lambda e^{i\theta}+\eta e^{-i\theta})|^2-(\eta-2\lambda)^2\sin^2[\theta]}}{\omega(2\lambda e^{i\theta}+\eta e^{-i\theta})} \tag{44}$$

with the condition

$$\omega^2(2\lambda+\eta)^2-[(2\lambda-\eta)^2+8\omega^2\eta\lambda]\sin^2[\theta] \geqslant 0. \tag{45}$$

The condition (45) guarantees that the right-hand side of (44) is a unit vector. It turns out that (45) delimits two wedge-like regions bounded by two straight lines, intersecting at the origin and symmetric about the coordinate axes, as shown in Fig. 3. Within the two wedge-like regions, the arcs of boundary curve of the neutral inclusion can be constructed through (44). Since these arcs can not form a closed curve, two straight lines, parallel to the uniaxial tensile stress, can be added to complete the closed boundary curve of neutral inclusion, see Fig. 3.

From (44), it is seen that $N(\bar{z}) = -N(z)$. This implies that the curves described by (44) are symmetric about the x-axis. Moreover, the right-hand side of (44) remains unchanged when z is replaced by $-z$ and, simultaneously, the square-root changes the sign. This implies that two curves described by the right-hand side of (44), obtained by taking the positive sign "+" and the negative sign "−" for the square-root and located in the right and the left half-planes, respectively, are symmetric with respect to the y-axis. Hence, as expected, the shape of neutral inclusion described by (44) is symmetric about the two coordinate axes. Thus, to examine the characters of the boundary curve, it is enough to consider the first quadrant, in which (44) gives

$$\cos[2N(z)] = \frac{(2\lambda-\eta)^2\sin^2[\theta]+(2\lambda+\eta)\cos[\theta]\sqrt{\omega^2(2\lambda+\eta)^2-[(2\lambda-\eta)^2+8\omega^2\eta\lambda]\sin^2[\theta]}}{\omega[(2\lambda-\eta)^2\sin^2[\theta]+(2\lambda+\eta)^2\cos^2[\theta]]} \tag{46}$$

and

$$\sin[2N(z)] = \frac{(2\lambda-\eta)(2\lambda+\eta)\sin[\theta]\cos[\theta]-(2\lambda-\eta)\sin[\theta]\sqrt{\omega^2(2\lambda+\eta)^2-[(2\lambda-\eta)^2+8\omega^2\eta\lambda]\sin^2[\theta]}}{\omega[(2\lambda-\eta)^2\sin^2[\theta]+(2\lambda+\eta)^2\cos^2[\theta]]}. \tag{47}$$

It follows from (46) that $\cos[2N(z)] > 0$. Furthermore, assuming that $2\lambda-\eta > 0$, then, in view of the fact that

$$(1-\omega^2)(2\lambda-\eta)\sin[\theta][(2\lambda+\eta)^2-8\lambda\eta\sin^2[\theta]] > 0$$

we find from (47) that $\sin[2N(z)] > 0$. These results imply that $\pi/4 > N(z) \geqslant 0$ when z is located in the first quadrant.

In view of (44), eqn (24) is now reduced to

$$\frac{16\cos[N(z)]}{(m(z)+n(z))} = \pm\sqrt{\omega^2|(2\lambda z+\eta\bar{z})|^2-(\eta-2\lambda)^2y^2}+(2\lambda+\eta)x \tag{48}$$

where "±" takes the same sign as in (44). It is readily seen that the sum $(m(z)+n(z))$ determined by (48) is positive provided that $2\lambda+\eta > 0$. Once $(m+n)$ is obtained, since the ratio $(m-n)/(m+n)$ is a known constant ω, the interface parameters $m(z)$ and $n(z)$ can be given by

$$m(z) = \frac{1+\omega}{2}(m+n), \quad n(z) = \frac{1-\omega}{2}(m+n). \tag{49}$$

Obviously, m and n are non-negative if $\omega^2 < 1$, and m is bigger than n if $\omega > 0$.

In summary, when $2\lambda \pm \eta > 0$, the interface design of a neutral inclusion under uniaxial tension can be implemented using a single adhesive layer. Once the adhesive material is chosen, the ratio $(m-n)/(m+n)$ is a known material constant (between zero and unity). The shape of the neutral inclusion can be constructed based on the equations (44–47). Next, the thickness of the adhesive layer can be determined in such a way that the corresponding sum $(m(z)+n(z))$ satisfies eqn (48). It is noted that the shape of neutral inclusion is determined by the material constants λ, η and ω. In particular, the length-to-width ratio of a neutral inclusion is bounded from below by the slope of two intersecting symmetric straight lines given by (45), see Fig. 3.

Finally, the only cases in which the two wedge-like regions (see Fig. 3) spread over the whole plane are: "$\eta - 2\lambda = 0$" and "$\omega = 1$". If $\eta = 2\lambda$, it follows from (44) and (48) that

$$e^{2iN(z)} = 1, \quad \frac{8\cos[N(z)]}{m(z)+n(z)} = 2\lambda[1+\omega]x. \tag{50}$$

Hence, the neutral inclusion is a rectangle. On the other hand, when $\omega = 1$, eqns (44) and (48) give

$$e^{2iN(z)} = 1, \quad \frac{8\cos[N(z)]}{(m(z)+n(z))} = (2\lambda+\eta)x \tag{51}$$

and the neutral inclusion is also a rectangle. Therefore, in these limit cases, the shape of neutral inclusion, described in Fig. 3, degenerates to a rectangle.

4. CONCLUSIONS

The model of imperfect interface is applied to the design of a single neutral elastic inclusion in plane and anti-plane deformations. Among the other results, it is found that the circular inclusion is the only neutral inclusion in each of the following cases: (1) the inclusion with a homogeneous interface ($h(z)$ is constant) in anti-plane shear, and (2) the equal-biaxial tension in plane deformations and the interface parameters are restricted such that the ratio $(m-n)/(m+n)$ is a given constant along the interface. In addition, since the interface with constant ratio $[m(z)-n(z)]/[m(z)+n(z)]$ can be easily realized using a single adhesive material, the design of a neutral elastic inclusion under uniaxial tension is studied in detail with such a restriction condition. The results obtained affirm the feasibility of designing a neutral elastic inclusion in many typical cases. In particular, the interface designed by the present method is independent on the magnitude of the prescribed stress field.

All interface parameters appearing in the paper are restricted to be non-negative. This leads to the basic restriction condition for the existence of a neutral elastic inclusion with the present method. This condition can be usually expressed in terms of the two bi-material constants λ and η, defined by (28). Roughly speaking, it requires that the inclusion is "harder" than the surrounding elastic body such that the deformation mismatch between the inclusion and the elastic body is compatible with the prescribed uniform stress field.

Finally, it is stated that the design of a neutral elastic inclusion can also be achieved using the method of eigen-strains. For example, for a circular inclusion with a perfectly bonded interface in plane deformations, it can be proven that the constant eigen-strains $\{\varepsilon^*_{xx}, \varepsilon^*_{xy}, \varepsilon^*_{yy}\}$ given by

$$\varepsilon_{xx}^* + \varepsilon_{yy}^* = \eta A, \quad (\varepsilon_{xx}^* - \varepsilon_{yy}^*) - i\varepsilon_{xy}^* + \lambda B = 0 \tag{52}$$

make the circular inclusion neutral under the uniform stress field characterized by the constants A and B. However, the design method based on the eigen-strains is essentially dependent on the magnitude of prescribed stress field (this is readily seen from (52)). As a result, the neutral inclusion designed by this method will cause non-uniform stress disturbance both inside and outside the inclusion when the magnitude of the prescribed stress field changes or reduces to zero. This undesirable property of the eigen-strain method seriously limits its application to many practical cases.

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